

A NOTE ON ABSOLUTELY SUMMING MULTILINEAR OPERATORS

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ABSTRACT. We obtain some optimal estimates for multilinear forms on ℓ_p spaces.

1. INTRODUCTION

A result due to Zaldueño ([7, Corollary 1]) asserts that if $m \geq 2$ is a positive integer and $p > m$, then

$$(1.1) \quad \left(\sum_{j=1}^n |T(e_j, \dots, e_j)|^{\frac{p}{p-m}} \right)^{\frac{p-m}{p}} \leq \|T\|$$

for all m -linear forms $T : \ell_p^n \times \dots \times \ell_p^n \rightarrow \mathbb{K}$ (here and henceforth $\mathbb{K} = \mathbb{R}$ or \mathbb{C}) and all positive integers n , and the exponent $\frac{p}{p-m}$ is optimal. In this note we investigate what happens if $1 < p \leq m$ and what happens if we still consider $p > m$ but with exponents $s \neq \frac{p}{p-m}$. Our results are shown to be optimal, except for the case $1 < p \leq 2$. Our result is the following:

Theorem 1. *Let $m \geq 2$ be a positive integer.*

(a) *If $s \geq 1$ and $p > m$, then*

$$(1.2) \quad \left(\sum_{j=1}^n |T(e_j, \dots, e_j)|^s \right)^{\frac{1}{s}} \leq \|T\| n^{\max\{\frac{m}{p} + \frac{1}{s} - 1, 0\}}$$

for all m -linear forms $T : \ell_p^n \times \dots \times \ell_p^n \rightarrow \mathbb{K}$ and all positive integers n , and the exponent is optimal.

(b) *If $s \geq 1$ and $2 \leq p \leq m$, then*

$$\left(\sum_{j=1}^n |T(e_j, \dots, e_j)|^s \right)^{\frac{1}{s}} \leq \|T\|$$

for all m -linear forms $T : \ell_p^n \times \dots \times \ell_p^n \rightarrow \mathbb{K}$ and all positive integers n , and the result is optimal.

(c) *If $s \geq \frac{2}{m}$ and $1 < p < 2$, then*

$$\left(\sum_{j=1}^n |T(e_j, \dots, e_j)|^s \right)^{\frac{1}{s}} \leq \|T\| n^{\frac{2ms+2p-spm}{2sp}}$$

for all m -linear forms $T : \ell_p^n \times \dots \times \ell_p^n \rightarrow \mathbb{K}$ and all positive integers n .

Remark 1. *For sums with multiple indices a similar result was recently obtained in [1].*

2. Proof

Proof of (a). From now on, $T : \ell_p^n \times \dots \times \ell_p^n \rightarrow \mathbb{K}$ is an m -linear form. If $s < \frac{p}{p-m}$, let x be such that

$$\frac{1}{s} = \frac{1}{p/(p-m)} + \frac{1}{x}.$$

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D. Pellegrino is supported by CNPq.

Using Hölder's inequality we have

$$\begin{aligned}
& \left(\sum_{j=1}^n |T(e_j, \dots, e_j)|^s \right)^{\frac{1}{s}} \\
& \leq \left(\sum_{j=1}^n |T(e_j, \dots, e_j)|^{\frac{p}{p-m}} \right)^{\frac{p-m}{p}} \cdot \left(\sum_{j=1}^n |1|^x \right)^{\frac{1}{x}} \\
& = \left(\sum_{j=1}^n |T(e_j, \dots, e_j)|^{\frac{p}{p-m}} \right)^{\frac{p-m}{p}} \cdot n^{\frac{1}{x}} \\
& \leq \|T\| \cdot n^{\frac{1}{s} - \frac{p-m}{p}} \\
& = \|T\| \cdot n^{\max\{\frac{m}{p} + \frac{1}{s} - 1, 0\}}
\end{aligned}$$

Now suppose that $s \geq \frac{p}{p-m}$. From the inclusion theorem for ℓ_p spaces we have, invoking Zaldueño's estimate (1.1)

$$\begin{aligned}
\left(\sum_{j=1}^n |T(e_j, \dots, e_j)|^s \right)^{\frac{1}{s}} & \leq \left(\sum_{i_1, \dots, i_m=1}^n |T(e_{i_1}, \dots, e_{i_m})|^{\frac{p}{p-m}} \right)^{\frac{p-m}{p}} \\
& \leq \|T\| \\
& = \|T\| \cdot n^{\max\{\frac{m}{p} + \frac{1}{s} - 1, 0\}}.
\end{aligned}$$

Now we prove the optimality. The optimality of the case $s \geq \frac{p}{p-m}$ is obvious. We just need to consider the case $s < \frac{p}{p-m}$. Consider $A : \ell_p^n \times \dots \times \ell_p^n \rightarrow \mathbb{K}$ (this idea we have borrowed from [3]) given by

$$A(x^{(1)}, \dots, x^{(m)}) = \sum_{j=1}^n x_j^{(1)} \dots x_j^{(m)}.$$

Note that, from the Hölder inequality we obtain

$$\|A\| \leq n^{\frac{p-m}{p}}$$

and thus, if the inequality (1.2) holds with n^t , we would have

$$n^{\frac{1}{s}} \leq C n^{\frac{p-m}{p}} n^t$$

and hence

$$t \geq \frac{m}{p} + \frac{1}{s} - 1.$$

Proof of (b). For $p \geq 2$, since ℓ_p has cotype p with cotype constant 1 (the fact that the cotype constant is 1 can be seen in [6, Lemma 2.3] or [5, page 29]), we know from ([2]) that every continuous m -linear form $T : \ell_p \times \dots \times \ell_p \rightarrow \mathbb{K}$ is absolutely $(\frac{p}{m}; 1, \dots, 1)$ -summing and the summing norm of T coincides with $\|T\|$. From the inclusion theorem for absolutely summing multilinear operators (see [4, Proposition 3.5] or [5, Proposition 3.3]) these forms are also absolutely $(1; p^*, \dots, p^*)$ -summing and, *a fortiori*, absolutely $(s; p^*, \dots, p^*)$ -summing regardless of the $s \geq 1$. So we conclude that

$$\begin{aligned}
& \left(\sum_{j=1}^n |T(e_j, \dots, e_j)|^s \right)^{\frac{1}{s}} \\
& \leq \|T\| \left(\sup_{\varphi \in B_{(\ell_p^n)^*}} \sum_{j=1}^n |\varphi(e_j)|^{p^*} \right)^{\frac{m}{p^*}} \\
& = \|T\|.
\end{aligned}$$

The optimality is immediate since we cannot have anything better than n^0 .

Proof of (c). For $1 < p < 2$, since ℓ_p has cotype 2 with cotype constant 1 (again, the information that the cotype constant is 1 can be seen in [6, Lemma 2.3] or [5, page 29]), we conclude from ([2]) that every continuous m -linear form $T : \ell_p \times \cdots \times \ell_p \rightarrow \mathbb{K}$ is absolutely $(\frac{2}{m}; 1, \dots, 1)$ -summing and, moreover, the respective absolutely summing norm coincides with the sup norm. From the inclusion theorem (see [4, Proposition 3.5] or [5, Prop. 3.3]) for absolutely summing multilinear operators, these forms are also absolutely $(s; \frac{2sm}{sm+2}, \dots, \frac{2sm}{sm+2})$ -summing. Since $p < 2$ it is easy to check that

$$(2.1) \quad \frac{2sm}{sm+2} < p^*.$$

Using (2.1) we thus conclude that

$$\begin{aligned} & \left(\sum_{j=1}^n |T(e_j, \dots, e_j)|^s \right)^{\frac{1}{s}} \\ & \leq \|T\| \left[\left(\sup_{\varphi \in B_{(\ell_p^n)^*}} \sum_{j=1}^n |\varphi(e_j)|^{\frac{2sm}{sm+2}} \right)^{\frac{sm+2}{2sm}} \right]^m \\ & = \|T\| \left(\sum_{j=1}^n \left(n^{\frac{-1}{p^*}} \right)^{\frac{2sm}{sm+2}} \right)^{\frac{sm+2}{2s}} \\ & = \|T\| \left(n \cdot \left(n^{\frac{1}{p}-1} \right)^{\frac{2sm}{sm+2}} \right)^{\frac{sm+2}{2s}} \\ & = \|T\| n^{\frac{sm+2}{2s} + \frac{m}{p} - m} \\ & = \|T\| n^{\frac{spm+2p+2ms-2spm}{2sp}} \\ & = \|T\| n^{\frac{2ms+2p-spm}{2sp}}. \end{aligned}$$

The authors were kindly informed by Daniel Galicer and Pilar Rueda, separately and in a private communication, and using different approaches, that the final solution to this problem is the following:

Theorem 2. *Let m, n be positive integers, $p_1, \dots, p_m \geq 1$ and $s > 0$. Let $C := C(m, n, p_1, \dots, p_m, s)$ be the best constant that fulfils the following inequality: for every m -linear form $A : \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \rightarrow \mathbb{K}$ we have*

$$\left(\sum_{j=1}^n |A(e_j, \dots, e_j)|^s \right)^{1/s} \leq C \|A\|_{\mathcal{L}(\ell_{p_1}^n, \dots, \ell_{p_m}^n)}.$$

The value of C is exactly
$$\begin{cases} (a) & n^{1/s} & \text{if } \frac{1}{p_1} + \cdots + \frac{1}{p_m} \geq 1, \\ (b) & n^{\max\{\frac{1}{p_1} + \cdots + \frac{1}{p_m} + \frac{1}{s} - 1, 0\}} & \text{if } \frac{1}{p_1} + \cdots + \frac{1}{p_m} \leq 1. \end{cases}$$

Remark 2. *As mentioned above, this final result is due to Daniel Galicer and Pilar Rueda (independently, and using different approaches), so the present note will not be submitted for publication.*

REFERENCES

- [1] G. Araújo, D. Pellegrino, Optimal Hardy–Littlewood type inequalities for m -linear forms on ℓ_p spaces with $1 \leq p \leq m$. *Archiv der Math.* 105 (2015), 285–295.
- [2] G. Botelho, Cotype and absolutely summing multilinear mappings and homogeneous polynomials. *Proc. Royal Irish Acad.* 97 (1997), 145–153.
- [3] V. Dimant and P. Sevilla-Peris, Summation of coefficients of polynomials on ℓ_p spaces, arXiv:1309.6063v1 [math.FA], to appear in *Publ. Mat.*
- [4] M.C. Matos, On multilinear mappings of nuclear type. *Rev. Mat. Univ. Complut. Madrid* 6 (1993), no. 1, 61–81.
- [5] D. Pérez-García, Operadores multilineales absolutamente sumantes, Thesis, Universidad Complutense de Madrid, 2003.
- [6] A. Tonge, Equivalence constants for matrix norms: a problem of Goldberg, *Linear Algebra Appl.* 306 (2000), 1–13.

- [7] I. Zaluendo, An estimate for multilinear forms on ℓ_p spaces. Proc. Royal Irish Acad. 93 (1993), 137–142.

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